

IS THERE ANYTHING NON-CLASSICAL?

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Abstract

It is argued that quantum logic and quantum probability theory are fascinating mathematical theories but without any relevance to our real world.

1 Introduction

At the beginning of this century, which is sometimes called the century of physics, we were presented with two revolutionary physical theories: the theory of relativity and the quantum theory. The common feature of these theories (and, perhaps, of the whole physics in this century) is their *counterintuitive* character. It would be beyond the scope of this paper to analyze the intellectual background at the turn of the century that made scientists so much attracted by everything against intuition. No doubt, in the twenties century physics “the logical basis is getting farther and farther from the empirical data, and the mental way leading to theorems directly related to the empirical observations is getting more and more long and hard” (Einstein, 1938). I believe, however, that in quantum mechanics we swung to the other extreme by enforcing counterintuitive abstractions, instead of struggling for real explanations.

The target of my critique is the *quantum probability theory* and *quantum logic*. The story begins with von Neumann’s recognition¹, that quantum mechanics *can be regarded* as a kind of probability theory defined over the subspace lattice $L(H)$ of a Hilbert space H . This recognition was confirmed by the *Gleason theorem*:

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¹This idea first appeared in Neumann 1932. One can find it in a more explicit and somewhat different form in Birkhoff and Neumann 1936.

Definition 1 A non negative real function μ on $L(H)$ is called a probability measure if $\mu(H) = 1$ and if whenever E_1, E_2, \dots are pairwise orthogonal subspaces, and $E = \bigvee_{i=1}^{\infty} E_i$, then $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$.

Theorem 1 (Gleason 1957) If H is a real or complex Hilbert space of dimension greater than 2, and μ is a probability measure on $L(H)$, then there exists a density operator W on H , such that $(\forall E \in L(H)) [\mu(E) = \text{tr}(WE)]$.²

Formally, the intersection and the (closed) linear union of subspaces play the role of *conjunction* and *disjunction* in an underlying *event lattice* of a *probability theory*. In spite of some difficulties, it is commonly accepted that, for example, a conjunction $A \wedge B$, represented by the intersection of the corresponding subspaces, $A \cap B$, corresponds to an event, which is nothing else, but the *joint occurrence* of events A and B . However, we have, as we will see it soon, many difficulties with such an interpretation! While everything is clear from the mathematical point of view, the *physical* meaning assigned to the elements of the subspace lattice and to the lattice operations is far from obvious.

The quantum probability — quantum logic approach is based on the conviction that there are phenomena described by quantum mechanics which cannot be accommodated in the classical Kolmogorov theory of probability. The majority of authors are shearing this conviction: Jauch (1968), Bub (1974), Putnam (1974), Piron (1976), Mittelstaedt (1978), Beltrametti and Cassinelli (1981), Gudder (1988), Pitowsky (1989), Kümmerer and Maassen (1996) and many others. Due to Feynmann (1951) this opinion is quite common among the users of quantum mechanics but who do not care to much with foundational questions.

There has been a serious critique against this approach. The first paper pointing out the pitfalls of quantum logic was published by Strauss (1937) a year after the famous Birkhoff and Neumann (1936). It is also worth mentioning a few papers of the last years arguing against non-classical probabilities: Ballentine (1989), Costantini (1992), Szabó (1995a,b), Gill (1996). It seems to me that quantum probabilists completely ignore the serious pitfalls pointed out by these authors. Bell expressed quite a similar disappointment:

Why did such serious people take so seriously axioms which now seem so arbitrary? I suspect that they were misled by the pernicious misuse of the word ‘measurement’ in contemporary theory. This word very strongly suggests the ascertaining of some preexisting property of some thing, any instrument involved playing a purely passive role. Quantum experiments are just not like that, as we learned especially from Bohr. The results have to be regarded as the joint product of ‘system’ and ‘apparatus,’ the complete experimental set-up. But the misuse of the word ‘measurement’ makes it easy to forget this and

²The subspaces, the corresponding projectors and the corresponding events are denoted by the same letter.

then to expect that the ‘results of measurements’ should obey some simple logic in which the apparatus is not mentioned. The resulting difficulties soon show that any such logic is not ordinary logic. It is my impression that the whole vast subject of ‘Quantum Logic’ has arisen in this way from the misuse of a word. I am convinced that the word ‘measurement’ has now been so abused that the field would be significantly advanced by banning its use altogether, in favor for example of the word ‘experiment.’ (Bell 1987, p. 166.)

My aim is to show in this paper that, beyond their vagueness and counterintuitiveness, quantum logic and quantum probability theory are *needless and inadequate*, because there is nothing in reality described by these mathematical constructions. In Section 2 I will show that whenever we consider non-commuting element of $L(H)$ nonsensical “probabilities” can appear which can hardly be interpreted as relative frequencies of any events. In Section 3 two often quoted examples, the double slit experiment and the EPR experiment are analyzed. These examples are usually meant to illustrate why we need to create a new, non-classical theory of probability. In both cases, however, it will be shown that no need to supersede the Kolmogorov theory of probability. In Section 4 we will see how can quantum phenomena, in general, be accommodated in the classical Kolmogorov theory of probability.

2 Nonsensical probabilities

Sometimes quantum mechanics produces very strange values of “probabilities”. Let me give a simple example.

2.1 Example I

Consider a system described in a 2-dimensional Hilbert space H^2 . Let φ and ψ be two orthogonal unit vectors. Suppose that the system is in the pure state $W = P_\varphi$. Consider non-commuting elements of $L(H^2)$: let E_1 be the one-dimensional subspace spanned by $(\cos \theta)\varphi + (\sin \theta)\psi$ and E_2 the subspace spanned by $(\cos \theta)\varphi - (\sin \theta)\psi$. The intersection is $E_1 \cap E_2 = \emptyset$. The probabilities of the corresponding events are

$$\begin{aligned} p(E_1) &= \text{tr}(WE_1) = \cos^2 \theta, \\ p(E_2) &= \text{tr}(WE_2) = \cos^2 \theta, \\ p(E_1 \wedge E_2) &= \text{tr}(W(E_1 \cap E_2)) = 0. \end{aligned}$$

If θ is close to zero then, for example,

$$p(E_1) = 0.9, p(E_2) = 0.9, \text{ while } p(E_1 \wedge E_2) = 0.$$

It is, however, *impossible* to interpret similar numbers as *relative frequencies* of occurrences of E_1 , E_2 and $E_1 \wedge E_2$. One cannot classify the possible histories of the universe in such a way, that 90% of them contains event E_1 , 90% of them contains E_2 , but none of them contains both E_1 and E_2 .

It is easy to see, that probabilities cannot be interpreted as relative frequencies if the following inequality is violated:

$$p(E_1) + p(E_2) - p(E_1 \wedge E_2) \leq 1. \quad (1)$$

To better understand the significance of the above example it is necessary to make a few remarks:

- If E_1 and E_2 commute then inequality (1) is always satisfied.
- At first sight one could think that the difficulties related with conjunctions occur only if the system is in a special state. But, it is easy to see that *for each state* one can find elements of $L(H)$, for which inequality (1) is violated.
- From the point of view of violation of inequality (1) the *complementarity* of E_1 and E_2 is irrelevant. One can easily create a similar example in space H^3 , such that $E_1 \wedge E_2 \neq \emptyset$ but inequality (1) is violated. The violation of inequality (1) means a more serious difficulty than the problem of complementary observables.
- Equation

$$p(E_1 \vee E_2) = p(E_1) + p(E_2) - p(E_1 \wedge E_2) \quad (2)$$

implies inequality (1). Remarkably, von Neumann regarded (2) as a fundamental property of a probability measure, which should be required in the quantum case, too. (Cf. Rédei, forthcoming)

2.2 Does non-commutativity always bear the danger of nonsensical probabilities?

It is not surprising that the appearance of nonsensical probabilities is related with non-commutative projectors. I must, however, emphasize that the naturality of this fact does not justify at all the adherence to nonsensical probabilities.

Theorem 2 *Let E_1 and E_2 be non-commuting elements of $L(H)$. There exists a pure state Ψ for which the probabilities violate inequality (1):*

$$\langle \Psi, E_1 \Psi \rangle + \langle \Psi, E_2 \Psi \rangle - \langle \Psi, (E_1 \wedge E_2 \Psi) \rangle > 1.$$

Proof

Arbitrary E_1 and E_2 can be written in the following form:

$$\begin{aligned} E_1 &= (E_1 \wedge E_2) \vee A, \\ E_2 &= (E_1 \wedge E_2) \vee B, \end{aligned} \quad (3)$$

such that $A \perp E_1 \wedge E_2$ and $B \perp E_1 \wedge E_2$.

First we prove the following statements:

- (a) $A \neq \emptyset$ and $B \neq \emptyset$ and $A \neq B$.
- (b) $A \not\perp B$.

Indeed, if $A = \emptyset$ or $B = \emptyset$ or $A = B$ would hold then either $E_1 < E_2$ or $E_2 < E_1$, that would contradict to the assumed non-commutativity of E_1 and E_2 . For proving (b) we show that from $A \perp B$ also the commutativity of E_1 and E_2 would follow. Commutativity is equivalent with $E_1 = (E_1 \wedge E_2) \vee (E_1 \wedge E_2^\perp)$. Using (3) we have

$$\begin{aligned} &(E_1 \wedge E_2) \vee \left[((E_1 \wedge E_2) \vee A) \wedge ((E_1 \wedge E_2) \vee B)^\perp \right] \\ &= (E_1 \wedge E_2) \vee \left[((E_1 \wedge E_2) \vee A) \wedge ((E_1 \wedge E_2)^\perp \wedge B^\perp) \right] \\ &= (E_1 \wedge E_2) \vee \left[((E_1 \wedge E_2) \vee A) \wedge (E_1 \wedge E_2)^\perp \right] \wedge B^\perp \end{aligned} \quad (4)$$

Since $A \perp (E_1 \wedge E_2)$, the distributivity holds in the square brackets. Therefore we can continue (4) as follows:

$$\begin{aligned} &= (E_1 \wedge E_2) \vee (A \wedge B^\perp) \\ &= (E_1 \wedge E_2) \vee A = E_1 \end{aligned}$$

which proves (b).

Now, from (a) and (b) it follows that there exists at least one normalized vector $\Psi \in A$ such that $\Psi \notin B$. Such a Ψ is a state vector for which the inequality

$$\underbrace{\langle \Psi, E_1 \Psi \rangle}_1 + \underbrace{\langle \Psi, E_2 \Psi \rangle}_{>0} - \underbrace{\langle \Psi, (E_1 \wedge E_2) \Psi \rangle}_0 > 1 \quad (5)$$

holds. □

The strange meaning of (5) is obvious! If E_1 happens with certainty, how can E_2 occur without E_1 ?

Thus, our partial conclusion can be this: 1) $L(H)$ can hardly play the role of an “algebra of events” for a probability theory. 2) The number $tr(WE)$ cannot be interpreted as the “relative frequency” of an event.

3 Do we really need quantum probability theory and quantum logic?

3.1 Example II: The double slit experiment

Our next example is the double slit experiment which is often quoted in order to justify why we need quantum probability theory (Fig. 1).

Denote $p(A)$ the probability of that “the particle arrives at a given point of the screen, Q, when only slit 1 is open”. $p(B)$ denotes the similar probability for slit 2. In the experiment one finds that

$$p(A) + p(B) \neq p(A \vee B), \quad (6)$$

where $p(A \vee B)$ stands for the probability of “the particle arrives at point Q either through slit 1 or slit 2”. According to the usual interpretation the double slit experiment shows that “*the method of computing probabilities involving subatomic particles is different from that of classical probability theory*” (Gudder 1988, p. 57). Therefore we must, as the usual conclusion says, 1) change probabilities for complex amplitudes (Feynmann) or 2) give up the Boolean event lattice (Quantum Logic) and classical probability theory (Quantum Probability Theory).

Contrary to these conclusions, let me ask:

3.2 Why don't we analyze the double slit example more carefully?

There are two different ways in which we can correctly describe the double slit experiment within the framework of classical probability theory. In both cases, it is the precise usage of notions “event” and “disjunction” what makes the classical probability theory satisfactory, while the formula (6) is, as we will see it soon, based on the misuse of these notions.

Version I

We must precisely distinguish the following events:

- A: “Slit 1 is open and slit 2 is closed and the particle is detected at Q”
- B: “Slit 1 is closed and slit 2 is open and the particle is detected at Q”
- C: “Slit 1 is open and slit 2 is open and the particle is detected at Q”

Obviously,

$$A \vee B \neq C.$$

Consequently we are not surprised that

$$p(A \vee B) = p(A) + p(B) \neq p(C).$$

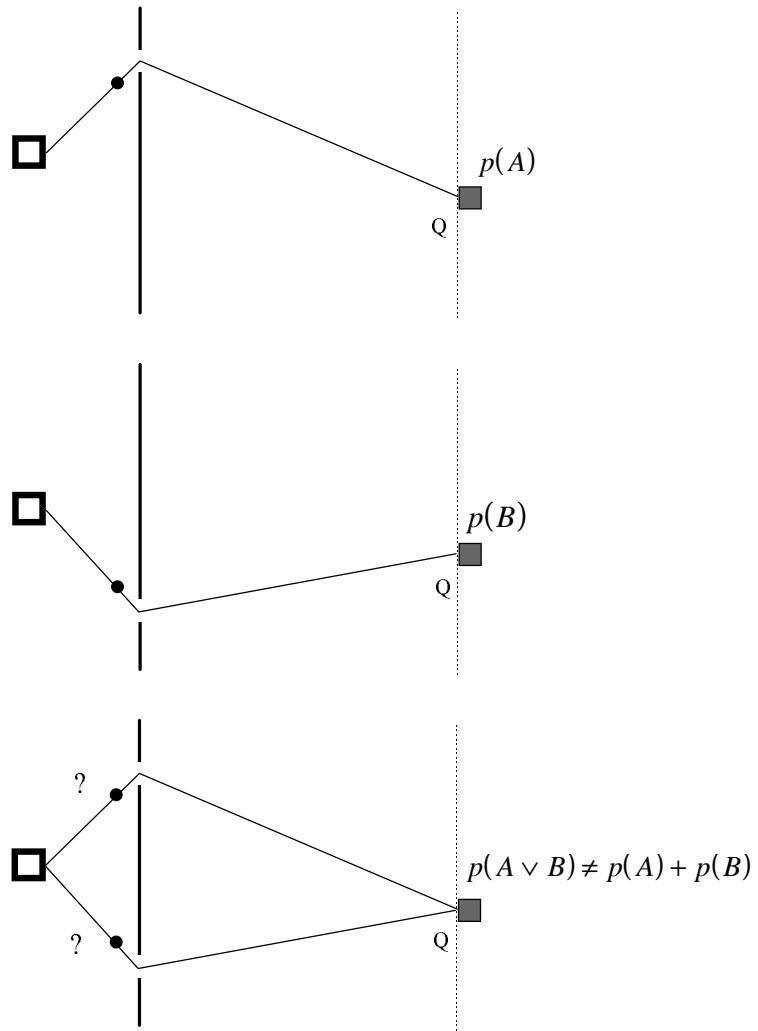


Figure 1: According to the usual interpretation the double slit experiment indicates that the rules of computing probabilities in quantum mechanics must be different from that of classical probability theory.

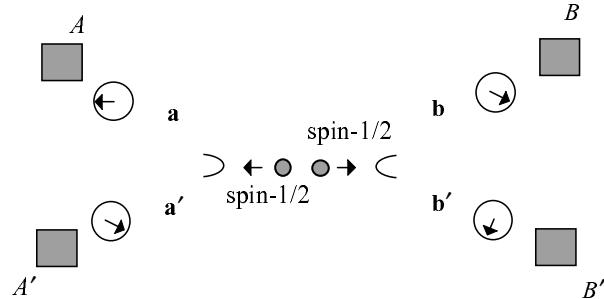


Figure 2: The Aspect experiment with spin- $\frac{1}{2}$ particles

That is, formula (6) is incorrect, consequently there is no violation of classical rules of probability calculation.

Version II

There is only one event:

D : “The particle is detected at Q ”

There are, however, different *conditions* under which the probabilities are understood. But the Kolmogorov axioms are meant to apply to probabilities belonging to *one common* system of conditions! Consequently, it does not mean a violation of the Kolmogorov axioms if

$$p_{1 \text{ is open; } 2 \text{ is closed}}(D) + p_{1 \text{ is closed; } 2 \text{ is open}}(D) \neq p_{1 \text{ is open; } 2 \text{ is open}}(D).$$

3.3 Example III: The EPR experiment

We have seen in the previous subsection that the double slit experiment does not prove the nonapplicability of Kolmogorov's classical theory of probability. It is true, however, that this example is not regarded as a serious one: it is rather used in the quantum mechanics text books only. In this subsection we are going to analyze the Einstein-Podolsky-Rosen experiment which is regarded as a crucial – empirically tested – situation providing probabilities which do not conform with the Kolmogorovian theory.

Consider an Aspect-type EPR experiment with spin- $\frac{1}{2}$ particles (Fig. 2). The four detectors detect the spin-up events. The two switches are making choice from sending the particles to the Stern-Gerlach magnets directed into different directions. The *observed* events are the followings:

- A : The “left particle has spin ‘up’ along direction \mathbf{a} ” detector beeps
- A' : The “left particle has spin ‘up’ along direction \mathbf{a}' ” detector beeps
- B : The “right particle has spin ‘up’ along direction \mathbf{b} ” detector beeps
- B' : The “right particle has spin ‘up’ along direction \mathbf{b}' ” detector beeps
- a : The left switch selects direction \mathbf{a}
- a' : The left switch selects direction \mathbf{a}'
- b : The right switch selects direction \mathbf{b}
- b' : The right switch selects direction \mathbf{b}'

For the probabilities of these events, in case of $\angle(\mathbf{a}, \mathbf{a}') = \angle(\mathbf{a}', \mathbf{b}) = \angle(\mathbf{a}, \mathbf{b}') = 120^\circ$ and $\angle(\mathbf{b}, \mathbf{a}') = 0$, we have

$$\begin{aligned}
p(A) = p(A') = p(B) = p(B') &= \frac{1}{4}, \\
p(a) = p(a') = p(b) = p(b') &= \frac{1}{2}, \\
p(A \wedge a) = p(A') &= \frac{1}{4}, \\
p(A' \wedge a') = p(A') &= \frac{1}{4}, \\
p(B \wedge b) = p(B) &= \frac{1}{4}, \\
p(B' \wedge b') = p(B') &= \frac{1}{4}, \\
p(A \wedge a') = p(A' \wedge a) = p(B \wedge b') = p(B' \wedge b) &= 0, \\
p(A \wedge B) = p(A \wedge B') = p(A' \wedge B') &= \frac{3}{32}, \\
p(A' \wedge B) &= 0, \\
p(a \wedge a) = p(b \wedge b') &= 0, \\
p(a \wedge b) = p(a \wedge b') = p(a' \wedge b) = p(a' \wedge b') &= \frac{1}{4}, \\
p(A \wedge b) = p(A \wedge b') = p(A' \wedge b) = p(A' \wedge b') \\
&= p(B \wedge a) = p(B \wedge a') = p(B' \wedge a) = p(B' \wedge a') &= \frac{1}{8}.
\end{aligned} \tag{7}$$

These statistical data *agree* with quantum mechanical results, in the following sense:

$$\begin{aligned}
\frac{p(A \wedge a)}{p(a)} &= \text{tr}(\hat{W}A) = \frac{p(A' \wedge a')}{p(a')} = \text{tr}(\hat{W}A') \\
&= \frac{p(B \wedge b)}{p(b)} = \text{tr}(\hat{W}B) = \frac{p(B' \wedge b')}{p(b')} = \text{tr}(\hat{W}B') = \frac{1}{2}, \\
\frac{p(A \wedge B \wedge a \wedge b)}{p(a \wedge b)} &= \frac{p(A \wedge B)}{p(a \wedge b)} = \text{tr}(\hat{W}AB)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sin^2 \angle(\mathbf{a}, \mathbf{b}) = \frac{3}{8}, \\
\frac{p(A \wedge B' \wedge a \wedge b')}{p(a \wedge b')} &= \frac{p(A \wedge B')}{p(a \wedge b')} = \text{tr}(\hat{W}AB') \\
&= \frac{1}{2} \sin^2 \angle(\mathbf{a}, \mathbf{b}') = \frac{3}{8}, \\
\frac{p(A' \wedge B \wedge a' \wedge b)}{p(a' \wedge b)} &= \frac{p(A' \wedge B)}{p(a' \wedge b)} = \text{tr}(\hat{W}A'B) \\
&= \frac{1}{2} \sin^2 \angle(\mathbf{a}', \mathbf{b}) = 0, \\
\frac{p(A' \wedge B' \wedge a' \wedge b')}{p(a' \wedge b')} &= \frac{p(A' \wedge B')}{p(a' \wedge b')} = \text{tr}(\hat{W}A'B') \\
&= \frac{1}{2} \sin^2 \angle(\mathbf{a}', \mathbf{b}') = \frac{3}{8},
\end{aligned} \tag{8}$$

where the outcomes are identified with the following projectors

$$\begin{aligned}
A &= \hat{P}_{\text{span}\{\psi_{+\mathbf{a}} \otimes \psi_{+\mathbf{a}}, \psi_{+\mathbf{a}} \otimes \psi_{-\mathbf{a}}\}}, \\
A' &= \hat{P}_{\text{span}\{\psi_{+\mathbf{a}'} \otimes \psi_{+\mathbf{a}'}, \psi_{+\mathbf{a}'} \otimes \psi_{-\mathbf{a}'}\}}, \\
B &= \hat{P}_{\text{span}\{\psi_{-\mathbf{b}} \otimes \psi_{+\mathbf{b}}, \psi_{+\mathbf{b}} \otimes \psi_{+\mathbf{b}}\}}, \\
B' &= \hat{P}_{\text{span}\{\psi_{-\mathbf{b}'} \otimes \psi_{+\mathbf{b}'}, \psi_{+\mathbf{b}'} \otimes \psi_{+\mathbf{b}'}\}}
\end{aligned}$$

of the Hilbert space $H^2 \otimes H^2$. The state of the system is assumed to be represented by $\hat{W} = \hat{P}_{\Psi_s}$, where $\Psi_s = \frac{1}{\sqrt{2}}(\psi_{+\mathbf{a}} \otimes \psi_{-\mathbf{a}} - \psi_{-\mathbf{a}} \otimes \psi_{+\mathbf{a}})$.

The question we would like to answer is whether the above probabilities, measured in the Aspect experiment, can be accommodated in a Kolmogorovian probability model, or not.

3.4 The Pitowsky formalism

Pitowsky elaborated a convenient geometric language for the discussion of the problem whether empirically given probabilities are Kolmogorovian or not (Pitowsky, 1989).

Let S be a set of pairs of integers $S \subseteq \{\{i, j\} \mid 1 \leq i < j \leq n\}$. Denote by $R(n, S)$ the linear space of real vectors having a form like $(f_1, f_2, \dots, f_{ij}, \dots)$. For each $\varepsilon \in \{0, 1\}^n$, let u^ε be the following vector in $R(n, S)$:

$$\begin{aligned}
u^\varepsilon &= \varepsilon_i, & 1 \leq i \leq n, \\
u_{ij}^\varepsilon &= \varepsilon_i \varepsilon_j, & \{i, j\} \in S.
\end{aligned}$$

Definition 2 The classical correlation polytope $\mathcal{C}(n, S)$ is the closed convex hull in $R(n, S)$ of vectors $\{u^\varepsilon\}_{\varepsilon \in \{0,1\}^n}$:

$$\mathcal{C}(n, S) := \left\{ a \in R(n, S) \mid a = \sum_{\varepsilon \in \{0,1\}^n} \lambda_\varepsilon u^\varepsilon, \text{ where } \lambda_\varepsilon \geq 0 \text{ and } \sum_{\varepsilon \in \{0,1\}^n} \lambda_\varepsilon = 1 \right\}.$$

Consider now events A_1, A_2, \dots, A_n and some of their conjunctions $A_i \wedge A_j$ ($\{i, j\} \in S$). Assume that we know their probabilities from which we can form a so called *correlation vector*:

$$\begin{aligned} \mathbf{p} &= (p_1, p_2, \dots, p_n, \dots, p_{ij}, \dots) \\ &= (p(A_1), p(A_2), \dots, p(A_n), \dots, p(A_i \wedge A_j), \dots) \in R(n, S). \end{aligned}$$

Definition 3 We will then say that \mathbf{p} has a Kolmogorovian representation if there exist a Kolmogorovian probability space (Ω, Σ, μ) and measurable subsets

$$X_{A_1}, X_{A_2}, \dots, X_{A_n} \in \Sigma,$$

such that

$$\begin{aligned} p_i &= \mu(X_{A_i}), & 1 \leq i \leq n, \\ p_{ij} &= \mu(X_{A_i} \cap X_{A_j}), & \{i, j\} \in S. \end{aligned}$$

Pitowsky's theorem tells us the necessary and sufficient condition a correlation vector must satisfy in order to be Kolmogorovian.

Theorem 3 (Pitowsky, 1989) A correlation vector

$$\mathbf{p} = (p_1, p_2, \dots, p_n, \dots, p_{ij}, \dots)$$

has a Kolmogorovian representation if and only if $\mathbf{p} \in \mathcal{C}(n, S)$.

In case $n = 4$ and $S = S_4 = \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$ the condition $\mathbf{p} \in \mathcal{C}(n, S)$ is equivalent with the following inequalities:

$$\begin{aligned} 0 &\leq p_{ij} \leq p_i \leq 1, \\ 0 &\leq p_{ij} \leq p_j \leq 1, \quad i = 1, 2 \quad j = 3, 4 \\ p_i + p_j - p_{ij} &\leq 1, \\ -1 &\leq p_{13} + p_{14} + p_{24} - p_{23} - p_1 - p_4 \leq 0, \\ -1 &\leq p_{23} + p_{24} + p_{14} - p_{13} - p_2 - p_4 \leq 0, \\ -1 &\leq p_{14} + p_{13} + p_{23} - p_{24} - p_1 - p_3 \leq 0, \\ -1 &\leq p_{24} + p_{23} + p_{13} - p_{14} - p_2 - p_3 \leq 0. \end{aligned} \tag{9}$$

(9) reminds us the well known *Clauser-Horne inequalities* (Clauser and Shimony 1978).³

Let us now apply inequalities (9) to the Aspect experiment in the usual way: Let

$$p_1 = \text{tr}(\hat{W}A), p_2 = \text{tr}(\hat{W}A'), p_3 = \text{tr}(\hat{W}B), p_4 = \text{tr}(\hat{W}B'),$$

$$p_{13} = \text{tr}(\hat{W}AB), p_{14} = \text{tr}(\hat{W}AB'), p_{23} = \text{tr}(\hat{W}A'B), p_{24} = \text{tr}(\hat{W}A'B'),$$

the values of which are given in (8). Substituting these values into the last inequality of (9) we find that

$$\mathbf{p} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, 0, \frac{3}{8} \right) \notin \mathcal{C}(n, S). \quad (10)$$

Thus, we can draw the *usual* conclusion: *the probabilities observed in the Aspect experiment have no Kolmogorovian representation.*

However, as I pointed out in my (1995a,b), a closer analysis yields a different conclusion! On the basis of particular examples I formulated the following hypothesis (Cf. 1995a). There is a “Kolmogorovian Censorship” in the real world: We never encounter “naked” quantum probabilities in reality. A correlation vector consisting of empirically testable probabilities is always a product

$$(p_1 \dots p_n \dots p_{ij} \dots) = (\pi_1 \dots \pi_n \dots \pi_{ij} \dots) \cdot (\tilde{p}_1 \dots \tilde{p}_n \dots \tilde{p}_{ij} \dots)$$

$$= (\pi_1 p_1 \dots \pi_n p_n \dots \pi_{ij} p_{ij} \dots),$$

where $(\pi_1 \dots \pi_n \dots \pi_{ij} \dots)$ are quantum probabilities and $(\tilde{p}_1 \dots \tilde{p}_n \dots \tilde{p}_{ij} \dots)$ are classical probabilities with which the corresponding measurements happen to be performed. The hypothesis says that such a product is always classical. (From the pure mathematical point of view, a product of a quantum and a classical correlation vector is not necessarily classical.) One can prove such a theorem within the framework of quite general assumptions (See Bana and Durt 1996).

Again, we must notice that Pitowsky theorem and, consequently, inequalities (9) apply to probabilities understood under *one common* system of conditions. Thus, we make a serious mistake by substituting conditional probabilities (8) into inequalities (9), since these conditional probabilities belong to *different* conditions.

4 How to join probability models?

³There is, however, an important conceptual disagreement between (9) and the original Clauser-Horne inequalities, see Szabó (1995b).

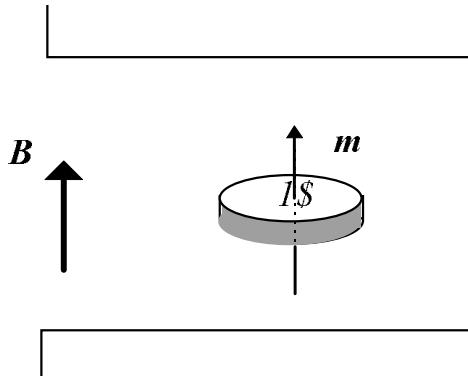


Figure 3: The probabilities of Heads (H) and Tails (T) are different if the magnetic field is on.

4.1 How to do it in the classical theory?

As we can see in the above examples, quantum mechanics produces Kolmogorovian probabilities belonging to different sets of conditions. The alleged impossibility to put these classical probability measures together into *one common* Kolmogorovian probability model is what urges us to cry for Quantum Probability Theory and Quantum Logic. But we *can* join these probability measures, if we do it in a correct way!

Before to seeing how we can do that, let us consider how this procedure goes in the classical theory of probability.

Let me take a simple example. We are tossing a coin which has a little magnetic momentum (Fig. 3). If the magnetic field is off, the probabilities are

$$\begin{aligned} p_{\text{off}}(\text{H}) &= 0.5, \\ p_{\text{off}}(\text{T}) &= 0.5. \end{aligned}$$

If the magnetic field is on, the probabilities are different:

$$\begin{aligned} p_{\text{on}}(\text{H}) &= 0.2, \\ p_{\text{on}}(\text{T}) &= 0.8. \end{aligned}$$

The event algebra \mathcal{A} is shown in Figure 4. Probability models $(\mathcal{A}, p_{\text{off}})$ and $(\mathcal{A}, p_{\text{on}})$ are, separately, Kolmogorovian. For example, they satisfy inequality (1):

$$p_{\text{off}}(\text{H}) + p_{\text{off}}(\text{T}) - p_{\text{off}}(\text{H} \wedge \text{T}) \leq 1, \quad (11)$$

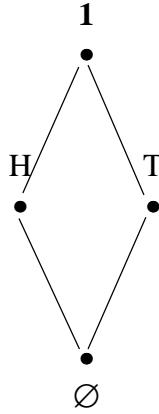


Figure 4: Algebra of events \mathcal{A}

and separately,

$$p_{\text{on}}(H) + p_{\text{on}}(T) - p_{\text{on}}(H \wedge T) \leq 1. \quad (12)$$

If we make the same mistake we did in the previous examples, and put these probabilities, belonging to different conditions, together into one formula prescribed for a Kolmogorovian probability theory, we find the same kind of “violation of the rules of classical probability theory”:

$$p_{\text{off}}(H) + p_{\text{on}}(T) - p_{\text{off}}(H \wedge T) = 0.5 + 0.8 > 1, \quad (13)$$

or

$$p_{\text{on}}(H) + p_{\text{off}}(T) = 0.2 + 0.5 \neq 1 = p_{\text{off}}(1) = p_{\text{off}}(H \vee T). \quad (14)$$

Consider now how to join probability models $(\mathcal{A}, p_{\text{off}})$ and $(\mathcal{A}, p_{\text{on}})$. In the classical probability theory we can join probabilities belonging to separate conditions only by enlarging the event algebra in such a way that it contains not only the original events but the “conditioning events”, too (Fig. 5)⁴. Of course, we can do that only if we know the probabilities of the conditioning events. In the example of question assume that $p(\text{OFF}) = 0.5$ and $p(\text{ON}) = 0.5$. So, the unified probability model is (\mathcal{A}', p) , where

$$\underline{p(1) = p(2) = p(9) = p(10)} = 0.25,$$

⁴I am grateful to Miltos Zissis for his warning that Fig. 5 was incorrect in a previous version of this paper

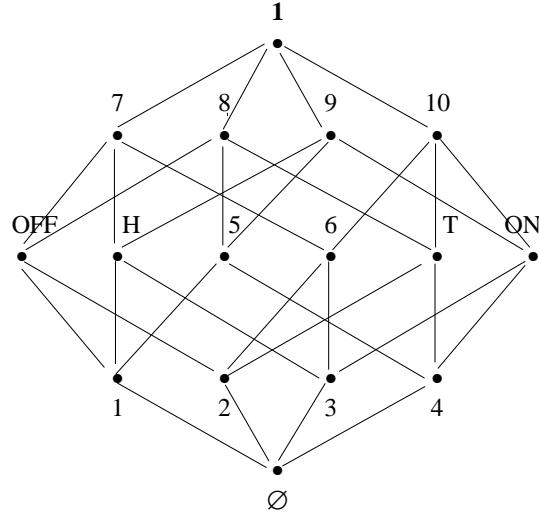


Figure 5: The unified algebra of events \mathcal{A}'

$$\begin{aligned}
 p(3) = p(8) &= 0.1, \\
 p(4) = p(7) &= 0.4, \\
 p(\text{OFF}) = p(\text{ON}) &= 0.5, \\
 p(\text{H}) = p(6) &= 0.35, \\
 p(\text{T}) = p(5) &= 0.65.
 \end{aligned} \tag{15}$$

The original probabilities are represented as conditional probabilities (defined by the Bayes law):

$$\begin{aligned}
 p_{\text{on}}(\text{H}) &= \frac{p(\text{H} \wedge \text{ON})}{p(\text{ON})} = \frac{p(3)}{p(\text{ON})} = \frac{0.1}{0.5} = 0.2, \\
 p_{\text{on}}(\text{T}) &= \frac{p(\text{T} \wedge \text{ON})}{p(\text{ON})} = \frac{p(4)}{p(\text{ON})} = \frac{0.4}{0.5} = 0.8, \\
 p_{\text{off}}(\text{H}) &= \frac{p(\text{H} \wedge \text{OFF})}{p(\text{OFF})} = \frac{p(1)}{p(\text{OFF})} = \frac{0.25}{0.5} = 0.5, \\
 p_{\text{off}}(\text{T}) &= \frac{p(\text{T} \wedge \text{OFF})}{p(\text{OFF})} = \frac{p(2)}{p(\text{OFF})} = \frac{0.25}{0.5} = 0.5.
 \end{aligned} \tag{16}$$

4.2 How to do it in the quantum theory?

Consider a quantum system described in Hilbert space H . The state of the system is represented by density operator W . Assume that there are N different measurements m_1, m_2, \dots, m_N one can carry out on the system. The corresponding observable-operators are denoted by $\hat{M}_1, \hat{M}_2, \dots, \hat{M}_N$. Let $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_N$ be the spectra of these operators. Introduce the following notation: each set of measurements $\{m_{i_1}, \dots, m_{i_s}\}$ will be identified with a vector $\eta \in \{0, 1\}^N$, such that

$$\eta_i = \begin{cases} 1 & \text{if } m_i \in \{m_{i_1}, \dots, m_{i_s}\}, \\ 0 & \text{if } m_i \notin \{m_{i_1}, \dots, m_{i_s}\}. \end{cases}$$

In this way the conditioning events can be represented in $2^{\{0,1\}^N}$. For instance, event “measurement m_i is performed” is represented by $\{\eta \mid \eta_i = 1\} \subset \{0, 1\}^N$, event “measurement m_i and measurement m_j are performed” corresponds to

$$\{\eta \mid \eta_i = 1\} \cap \{\eta \mid \eta_j = 1\},$$

etc.

Some of these measurements can be incompatible, in the sense that they cannot be simultaneously carried out. Assume that, according to the quantum theory, observables belonging to compatible measurements commute. For each set $\{m_{i_1}, m_{i_2}, \dots, m_{i_s}\} = \{m_i\}_{\eta_i=1}$ of *compatible* measurements the quantum state W determines a Kolmogorovian probability measure over the corresponding Borel sets, $\left(B\left(\bigtimes_{\eta_i=1} \mathcal{M}_i\right), \mu_\eta\right)$, where

$$\mu_\eta : (A_i)_{\eta_i=1} \in B\left(\bigtimes_{\eta_i=1} \mathcal{M}_i\right) \mapsto \text{tr} \left(W \prod_{\eta_i=1} A_i\right). \quad (17)$$

Now, how can we join these classical probability spaces into one common classical probability model? The method is known from the classical theory of probability. Quantum mechanics has nothing special from this point of view! That is, we need to enlarge the event algebra by the conditioning events and to define the joint probability measure over this larger algebra of events. In order to do that, we need to know the probabilities of conditioning events. The values of these probabilities are the matter of empirical facts. Although, the following assumption seems to be quite plausible:

Stipulation *There is a classical probability measure \tilde{p} on $2^{\{0,1\}^N}$, such that if $\tilde{p}(\{\eta\}) \neq 0$ then the corresponding set of operators $\{\hat{M}_i\}_{\eta_i=1}$ is commuting.*

Thus, my assertion is that classical probabilities (17) can be joint into one Kolmogorovian probability model:

Theorem 4 *There exists a Kolmogorovian probability space $(\mathcal{M}, B(\mathcal{M}), p)$ such that each conditioning event $E \in 2^{\{0,1\}^N}$ and each outcome event $(A_i)_{\eta_i=1}$ can be represented by an element of $B(\mathcal{M})$, denoted by X_E and $X_{(A_i)_{\eta_i=1}}$, respectively, and*

$$\tilde{p}(E) = p(X_E), \quad \forall E \in 2^{\{0,1\}^N},$$

$$\begin{aligned} \mu_\eta((A_i)_{\eta_i=1}) &= \text{tr}\left(W \prod_{\eta_i=1} A_i\right) & \forall \eta \in \{0,1\}^N \\ &= \frac{p(X_{(A_i)_{\eta_i=1}} \cap X_{\{\eta\}})}{p(X_{\{\eta\}})}, & \forall (A_i)_{\eta_i=1} \in \bigtimes_{\eta_i=1} \mathcal{M}_i. \end{aligned}$$

Proof The enlarged Boolean algebra of events can be constructed as a Boolean σ -algebra of Borel sets $B(\mathcal{M})$, where

$$\mathcal{M} = \bigcup_{\eta \in \{0,1\}^N} \bigtimes_{\eta_i=1}^{\text{disjoint}} \mathcal{M}_i. \quad (18)$$

An original outcome event $A_i \in B(\mathcal{M}_i)$ is represented by

$$\bigcup_{\substack{\eta \in \{0,1\}^N \\ \eta_i=1}}^{\text{disjoint}} \left(\bigtimes_{\substack{\eta_j=1 \\ j < i}} \mathcal{M}_j \right) \times A_i \times \left(\bigtimes_{\substack{\eta_k=1 \\ k > i}} \mathcal{M}_k \right).$$

A conditioning event $\{\eta\}$ can be identified with

$$\bigtimes_{\eta_i=1} \mathcal{M}_i.$$

Now, the joint classical probability model is $(\mathcal{M}, B(\mathcal{M}), p)$, where the probability measure p is generated by the following rule:

$$p : \left(\bigtimes_{\eta_i=1} A_i \right) \in B(\mathcal{M}) \mapsto \text{tr} \left(W \prod_{\eta_i=1} A_i \right) \cdot \tilde{p}(\eta). \quad (19)$$

It is worth mentioning that if η represents incompatible measurements then probability (19) is zero.

The original probabilities belonging to different particular condition are also reproduced as conditional probabilities: for example the probability of an outcome A_i given that measurement m_i is carried out is $\text{tr}(WA_i)$, and indeed,

$$\begin{aligned}
& \frac{p\left(\bigcup_{\substack{\eta \in \{0,1\}^N \\ \eta_i=1}}^{\text{disjoint}} \left(\bigtimes_{\substack{\eta_j=1 \\ j < i}} \mathcal{M}_j\right) \times A_i \times \left(\bigtimes_{\substack{\eta_k=1 \\ k > i}} \mathcal{M}_k\right)\right)}{p\left(\bigtimes_{\eta_i=1} \mathcal{M}_i\right)} \\
&= \frac{\sum_{\substack{\eta \in \{0,1\}^N \\ \eta_i=1}} \text{tr}\left(W\left(\prod_{\eta_j=1} \text{Id}_j\right) A_i\right) \tilde{p}(\eta)}{\sum_{\substack{\eta \in \{0,1\}^N \\ \eta_i=1}} \tilde{p}(\eta)} \quad (20) \\
&= \frac{\text{tr}\left(W A_i\right) \sum_{\substack{\eta \in \{0,1\}^N \\ \eta_i=1}} \tilde{p}(\eta)}{\sum_{\substack{\eta \in \{0,1\}^N \\ \eta_i=1}} \tilde{p}(\eta)} = \text{tr}(WA_i).
\end{aligned}$$

□

Let us apply this general method to the EPR experiment. We have 4 possible measurements a, a', b, b' . There are 8 different conditioning events symbolized with the 8 vectors $\eta \in \{0,1\}^4$. The probabilities of these conditioning events are

$$\begin{aligned}
\tilde{p}(0000) = \tilde{p}(1000) = \tilde{p}(0100) = \tilde{p}(0010) = \tilde{p}(0001) = \tilde{p}(1100) = \tilde{p}(0011) &= 0, \\
\tilde{p}(1010) = \tilde{p}(1001) = \tilde{p}(0110) = \tilde{p}(0101) &= \frac{1}{4}.
\end{aligned}$$

Each of the four observables has a spectrum consisting from two points “up” and “down”. Thus, the enlarged event algebra is the Boolean algebra generated by 81 elementary events $E \in \{\text{up}, \text{down}, \text{none}\}^4$. The joint probability measure is determined by the probabilities of the elementary events. There are 16 elementary events which have non-zero probability:

$$\begin{aligned}
& p(\text{up none up none}) = p(\text{up none none up}) \\
&= p(\text{none up none up}) = p(\text{down none down none}) \\
&= p(\text{down none none down}) = p(\text{none down none down}) = \frac{3}{32},
\end{aligned}$$

$$\begin{aligned}
& p(\text{up none down none}) = p(\text{up none none down}) \\
&= p(\text{none up none down}) = p(\text{down none up none}) \\
&= p(\text{down none none up}) = p(\text{none down none up}) = \frac{1}{32}, \\
& p(\text{none down up none}) = p(\text{none up down none}) = \frac{1}{8}.
\end{aligned}$$

To see that this is a consistent representation, let us check one of the probabilities in (8): event $A \wedge B$ is represented by subset

$$X_A \cap X_B = \left\{ (\text{up none up none}), (\text{up up up none}), (\text{up down up none}), (\text{up none up down}), (\text{up none up up}) \right\}.$$

The probability $p(X_A \cap X_B) = \frac{3}{32} + 0 + 0 + 0 + 0 = \frac{3}{32}$. Condition event $a \wedge b$ is represented by

$$X_a \cap X_b = \left\{ (\text{up none up none}), (\text{down none up none}), (\text{down none down none}), (\text{up none down none}), (\text{up up up up}), (\text{up up up none}), \dots (\text{down down down down}) \right\}$$

and

$$p(X_a \cap X_b) = \frac{3}{32} + \frac{1}{32} + \frac{3}{32} + \frac{1}{32} + 0 + 0 \dots + 0 = \frac{1}{4}.$$

As it is required, $X_A \cap X_B \subset X_a \cap X_b$. Consequently,

$$tr(WAB) = \frac{p(X_A \cap X_B \cap X_a \cap X_b)}{p(X_a \cap X_b)} = \frac{p(X_A \cap X_B)}{p(X_a \cap X_b)} = \frac{\frac{3}{32}}{\frac{1}{4}} = \frac{3}{8},$$

as it was expected.

Conclusions

The analysis of the above examples and Theorem 4 unanimously tell us that quantum mechanics, regarded as a physical theory about empirical facts of our world, does not demand to supersede the classical theory of probability. It is needless to do that and, what is more, any attempt at the empirical foundation of the quantum probability theory seems to be contradictory. Like it or not, quantum mechanics is connected with the empirical facts about the world, which it is supposed to be applied to, through relative frequencies. But those “probabilities” that are presented by the quantum probability theory can hardly be interpreted as relative frequencies of events. And whether we like it or not,

quantum logic is nothing else but an algebraic structure isomorphic with the algebra of events underlying the quantum probability theory. So, if quantum probability theory has nothing to do to reality then quantum logic is meaningless, too. Quantum logic and quantum probability theory remain fascinating mathematical theories but without any relevance to our real world.

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